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# Distribution tails of sample quantiles and subexponentiality

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**Abstract**

We show that subexponentiality is not sufficient to guarantee that the distribution tail of a sample quantile of an infinitely divisible process is equivalent to the "tail" of the same sample quantile under the corresponding Lévy measure. However, such an equivalence result is shown to hold under either an assumption of an appropriately slow tail decay or an assumption on the structure of the process. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Given a measurable real valued stochastic process $\mathbf{X}=(X(t),\ t\in T)$ defined on a probability space $(\Omega,\mathscr{F},P)$ and indexed by a parameter $t$ in a probability space $(T,\mathscr{T},m)$ and a $\rho\in[0,1)$ we define the sample quantile of $\mathbf{X}$ as

$$Q_\rho(\mathbf{X})=\inf\left\{x\in\mathbb{R}:\ \int_T \mathbf{1}_{\{X(t)\leqslant x\}}m(\mathrm{d}t)>\rho\right\}. \tag{1}$$

A sample quantile is a functional of a sample path of the process. By definition, $\mathbf{X}$ spends at least $100\rho\%$ of its "time" at or below $Q_\rho(\mathbf{X})$, and at least $100(1-\rho)\%$ of its "time" at or above $Q_\rho(\mathbf{X})$. The recent interest in the properties of sample quantiles originates, probably, with mathematical finance and "look-back" options. See Miura (1992) to learn more about "exotic" options, and Dassios (1995, 1996) and Embrechts et al. (1995) for some recent research on sample quantiles.

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Suppose that the process $\boldsymbol{X}$ is infinitely divisible, and given in the form

$$X(t)=\int_S f(t,s)M(\mathrm{d}s), \quad t\in T, \tag{2}$$

where $(S,\mathscr{A})$ is a measurable space and $M$ is an independently scattered infinitely divisible random measure on $(S,\mathscr{A})$ with Lévy measure $F$. We recall that $F$ is a $\sigma$-finite measure on $(S\times\mathbb{R},\mathscr{A}\times\mathscr{B})$, where $\mathscr{B}$ is the Borel $\sigma$-field on $\mathbb{R}$. The random measure $M$ can be regarded as a stochastic process $(M(A),\ A\in\mathscr{A}_0)$, where

$$\mathscr{A}_0=\left\{A\in\mathscr{A}:\ \int_A\int_{\mathbb{R}}\min(1,x^2)F(\mathrm{d}s,\mathrm{d}x)<\infty\right\}.$$

Moreover, for every $A\in\mathscr{A}_0$, $M(A)$ is a real infinitely divisible random variable with a one-dimensional Lévy measure $\eta$ given by $\eta(B)=F(A\times B)$. That is,

$$\begin{aligned}E\exp(\mathrm{i}\theta M(A))&=\exp\left\{\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta\tau(x))\eta(\mathrm{d}x)\right\}\\&=\exp\left\{\int_A\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta x}-1-\mathrm{i}\theta\tau(x))F(\mathrm{d}s,\mathrm{d}x)\right\},\end{aligned}\tag{3}$$

where $\tau(x)=x/(1+x^2)$.

The kernel $f(t,s)$ in Eq. (2) must have certain integrability properties for the stochastic integral to be well defined and the resulting stochastic process to be measurable. See Rajput and Rosiński (1989) for more details on infinitely divisible random measures and stochastic integrals with respect to such measures. In particular, we will assume that the kernel $f:S\times T\to\mathbb{R}$ is (jointly) measurable. See Appendix in Embrechts and Samorodnitsky (1995). We are interested in relating the tail distributional properties of a sample quantile $Q_\rho(\boldsymbol{X})$ to the corresponding properties of the quantiles of the kernel $f$ in (2) with respect to the Lévy measure $F$.

A nonnegative random variable $X$ is said to be *subexponential* (or to have a subexponential distribution) if $P(X>\lambda)>0$ for all $\lambda>0$, and

$$\lim_{\lambda\to\infty}\frac{P(X_1+X_2>\lambda)}{P(X>\lambda)}=2.$$

Here $X_1$ and $X_2$ are independent copies of $X$. A real valued random variable $X$ is called subexponential if its positive part $X_+$ is. The defining tail equivalence property of subexponential random variables extends to more complicated sums. Thus, if $X_1,X_2,\ldots$ are i.i.d. copies of a subexponential random variable $X$, then

$$\lim_{\lambda\to\infty}\frac{P(X_1+\cdots+X_n>\lambda)}{P(X>\lambda)}=n \tag{4}$$

for every $n\geqslant 1$, and

$$\lim_{\lambda\to\infty}\frac{P(X_1+\cdots+X_N>\lambda)}{P(X>\lambda)}=\gamma, \tag{5}$$

where $N$ is a mean $\gamma$ Poisson random variable independent of the sequence $X_1,X_2,\ldots$ . One of the consequences of Eq. (5) is the following. If $X$ is an infinitely divisible random variable with Lévy measure $\eta$ such that its tail $\eta(\lambda,\infty)\wedge 1$ is the distribution tail of a subexponential random variable, then

$$\lim_{\lambda\to\infty}\frac{P(X>\lambda)}{\eta(\lambda,\infty)}=1. \tag{6}$$

See Chover et al. (1973) and Embrechts et al. (1979).

On the level of stochastic processes, subexponentiality causes a phenomenon similar to Eq. (6) for *subadditive functionals* of the sample paths of the processes. Specifically, let $\boldsymbol{X}$ be an infinitely divisible process given by Eq. (2) and $\phi:\boldsymbol{R}^{\mathrm{T}}\to(-\infty,\infty]$ a measurable function satisfying $\phi(\boldsymbol{x}_1+\boldsymbol{x}_2)\leqslant\phi(\boldsymbol{x}_1)+\phi(\boldsymbol{x}_2)$ for every $\boldsymbol{x}_1,\boldsymbol{x}_2\in\boldsymbol{R}^{\mathrm{T}}$ (subadditivity property). One looks at the "tail" of functional $\phi$ of the kernel $\boldsymbol{f}(s)=(f(t,s), t\in T)$, $s\in S$ by defining $H(\lambda)=F\{(s,x)\in S\times\boldsymbol{R}\colon \phi(x\boldsymbol{f}(s))>\lambda\}$, $\lambda>0$. If $H(\lambda)\wedge 1$ is the distribution tail of a subexponential random variable, then (under a boundedness assumption),

$$\lim_{\lambda\to\infty}\frac{P(\phi(\boldsymbol{X})>\lambda)}{H(\lambda)}=1. \tag{7}$$

See Rosiński and Samorodnitsky (1993).

Our goal in this paper is to study to what extent (7) extends to sample quantiles of an infinitely divisible process. Formally, we would like to replace $\phi$ with $Q_\rho$ above. In this case the corresponding tail measure defined on the kernel $\boldsymbol{f}(s)$ is given by

$$H_\rho(\lambda)=F\{(s,x)\in S\times\mathbb{R}\colon Q_\rho(x\boldsymbol{f}(s))>\lambda\}. \tag{8}$$

Suppose that $H_\rho(\lambda)\wedge 1$ is the distribution tail of a subexponential random variable. Does this imply (perhaps, under appropriate boundedness assumptions) that

$$\lim_{\lambda\to\infty}\frac{P(Q_\rho(\boldsymbol{X})>\lambda)}{H_\rho(\lambda)}=1\,? \tag{9}$$

The answer is not supplied (at least, directly) by (7) because the sample quantile is not a subadditive functional of a process. However, Embrechts and Samorodnitsky (1995) gave an affirmative answer to the above question under more restrictive assumptions. The main assumption was that $H_\rho(\lambda)$ was regularly or slowly varying at infinity. Distributions with regularly varying tails form an important, but only a proper, subclass of all distributions with subexponential tails, and so the question whether (9) extends to the entire subexponential setting remained open.

The first task we accomplish in this paper is to demonstrate by means of an example that (9) does not extend to the entire subexponential setting, even under the same additional assumptions that were required in the regularly varying case. This is done in Section 2. We then show that (9) still holds if, in addition to subexponentiality, the tail $H_\rho(\lambda)$ is heavy enough (although not necessarily as heavy or as regular as a regularly varying tail). This result is presented in Section 3. Finally, in Section 4 we show that certain *structural* assumptions on the process will guarantee the tail equivalence (9) without any additional *rate of decay* assumptions as in Section 3. For example, it turns out that sample quantiles of both Lévy motions and integrated Lévy motions satisfy (9); note that in the financial context the former process is often viewed as the (log) price process, while the latter process is simply related to the average (log) price process.

## 2. Bad news: A counterexample

In this section we give an example of an infinitely divisible process such that $H_\rho(\lambda)$ is equivalent to a subexponential distribution tail, but (9) does not hold even under the additional assumptions imposed by Embrechts and Samorodnitsky (1995) in the regularly varying case.

Let us consider a modified Ornstein–Uhlenbeck process

$$X(t) = \int_0^t \mathrm{e}^{(s-t)} M(\mathrm{d}s), \tag{10}$$

$0 \leqslant t \leqslant 1$, where the Lévy measure $F$ of the infinitely divisible random measure $M$ is given by $F(\mathrm{d}s, \mathrm{d}x) = \mathrm{Leb}(\mathrm{d}s)\mu(\mathrm{d}x)$. Here Leb is the Lebesgue measure on $[0,1]$ and $\mu$ a one-dimensional Lévy measure. It is easy to see that (10) is a well defined measurable infinitely divisible process for any $\mu$ as above. Let $m = \mathrm{Leb}$ as well.

For a general process (2) one defines

$$H_*(\lambda) = F\left\{(s,x) \in S \times \mathbb{R}\colon |x| \operatorname*{ess\,sup}_{t\in T} |f(t,s)| > \lambda\right\}. \tag{11}$$

A simple substitution shows that for the process (10) we have

$$H_*(\lambda) = \mu((\lambda, \infty)), \tag{12}$$

and that

$$H_\rho(\lambda) = \rho\mu((\lambda \mathrm{e}^{1-\rho}, \infty)). \tag{13}$$

For our example we choose a $\mu$ supported on $[e, \infty)$ given by

$$\mu((\lambda,\infty)) = \exp\left(-\frac{\lambda}{\log\lambda}\right) \tag{14}$$

for $\lambda > e$. It is straightforward to check Eq. (14) is the tail of a subexponential random variable; see Pitman (1980). Since scaling does not affect subexponentiality, it follows immediately that $H_\rho$ is equivalent to the tail of a subexponential distribution as well. We claim that for this process

$$\lim_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{H_\rho(\lambda)} = \infty, \tag{15}$$

for any $\rho \in (0,1)$ while for any $\rho > 1 - \log 2$

$$\lim_{\lambda\to\infty} \frac{(H_*(\lambda))^2}{H_\rho(\lambda)} = 0. \tag{16}$$

Therefore, the conclusion (9) fails for our process for all $\rho \in (0,1)$. The reason we check Eq. (16) is that it is under this assumption that Eq. (9) was proved in Embrechts and Samorodnitsky (1995) in the regularly varying case. Moreover, the function $H_*(\lambda)$ will play an important role in the results of the next section.

We start with observing that it follows from Eqs. (12) and (13) that in our case

$$H_*(\lambda)=\exp\left(-\frac{\lambda}{\log\lambda}\right)$$

and

$$H_\rho(\lambda)=\rho\exp\left(-e^{(1-\rho)}\frac{\lambda}{\log\lambda+(1-\rho)}\right). \tag{17}$$

In particular, if $\rho>1-\log 2$ then Eq. (16) holds.

To check Eq. (15) we start with observing that we can represent the process (10) in the form

$$X(t)=e^{-t}\sum_{k=1}^{N(t)}e^{\Gamma_k}Y_k,\quad 0\leqslant t\leqslant 1, \tag{18}$$

where $(N(t),\ t\geqslant 0)$ is a unit rate Poisson process independent of a sequence of i.i.d. random variables $Y_1,Y_2,\ldots$ with a common distribution given by $P(Y_1>\lambda)=\mu((\lambda,\infty))$. Here $\Gamma_1,\Gamma_2,\ldots$ are the arrival times of the Poisson process $(N(t),\ t\geqslant 0)$. Let

$$L_x(\boldsymbol{X})=\int_0^1 \mathbf{1}_{\{X(t)>x\}}\,dt$$

be the sojourn time functional of the process. Observe that $L_x(\boldsymbol{X})>1-\rho$ for a $\rho\in(0,1)$ implies that $Q_\rho(\boldsymbol{X})>x$. Therefore, Eq. (15) will follow once we prove that

$$\lim_{\lambda\to\infty}\frac{P(L_\lambda(\boldsymbol{X})>1-\rho,\ N(1)=2)}{H_\rho(\lambda)}=\infty. \tag{19}$$

A simple geometric argument together with (18) shows that on the event $\{N(1)=2\}$ we have

$$\begin{aligned}L_\lambda(\boldsymbol{X})&=\min\{(\log Y_1-\log\lambda)^+,\Gamma_2-\Gamma_1\}\\&\quad+\min\{(\log(Y_1e^{\Gamma_1-\Gamma_2}+Y_2)-\log\lambda)^+,1-\Gamma_2\}.\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}&P(L_\lambda(\boldsymbol{X})>1-\rho,\ N(1)=2)\\&\quad\geqslant P(L_\lambda(\boldsymbol{X})>1-\rho,\ N(1)=2,\ L_\lambda(\boldsymbol{X})=(\Gamma_2-\Gamma_1)+(1-\Gamma_2))\\&\quad=P(Y_1e^{\Gamma_1-\Gamma_2}>\lambda,\ Y_1e^{\Gamma_1-1}+Y_2e^{\Gamma_2-1}>\lambda,\ \Gamma_1<\rho,\ \Gamma_2<1<\Gamma_3)\\&\quad=e^{-1}\int_{\substack{0<u<v<1\\u<\rho}}P(Y_1e^{u-v}>\lambda,\ Y_1e^{u-1}+Y_2e^{v-1}>\lambda)\,du\,dv.\end{aligned} \tag{21}$$

We now estimate the probability under the double integral in Eq. (21) for fixed $u$ and $v$. Denote $b=e^{u-1}$ and $c=e^{v-1}$. Then $0<b<c<1$ and the probability in question is

$$U(b,c,\lambda):=P(bc^{-1}Y_1>\lambda,\ bY_1+cY_2>\lambda)\geqslant P(Y_1>cb^{-1}\lambda)P(Y_2>(c^{-1}-1)\lambda)$$

because of the independence of $Y_1$ and $Y_2$. Now Eq. (14) implies that

$$U(b,c,\lambda)\geqslant \exp\left[\frac{\lambda(c^{-1}-1+cb^{-1})}{\log\lambda}B(b,c,\lambda)\right],$$

where

$$B(b,c,\lambda):=\frac{\log\lambda}{\log(\lambda(c^{-1}-1))}\frac{c^{-1}-1}{(c^{-1}-1+cb^{-1})}+\frac{\log\lambda}{\log(\lambda cb^{-1})}\frac{cb^{-1}}{(c^{-1}-1+cb^{-1})}.$$

For every $\delta\in(0,1)$, $B(b,c,\lambda)\to 1$ as $\lambda\to\infty$, uniformly on the set $\{0<\delta\leqslant b\leqslant c\leqslant 1-\delta\}$. We conclude that for every $\delta\in(0,1)$ and $\varepsilon>0$ there exists a $\lambda_0=\lambda_0(\varepsilon,\delta)$ such that

$$U(b,c,\lambda)\geqslant \exp\left[-\left(\frac{\lambda(c^{-1}-1+cb^{-1})}{\log\lambda}(1+\varepsilon)\right)\right]$$

for all $\lambda>\lambda_0$ and all $0<\delta\leqslant b\leqslant c\leqslant 1-\delta$.

For $0<u<v<1$ we define

$$V(u,v)=\mathrm{e}^{1-v}-1+\mathrm{e}^{v-u}.$$

Recalling the definition of $b$ and $c$ above we conclude immediately that for every $\delta\in(0,1)$ and $\varepsilon>0$ there exists a $\lambda_0=\lambda_0(\varepsilon,\delta)$ such that

$$U(b,c,\lambda)\geqslant \exp\left[-\left(\frac{\lambda V(u,v)}{\log\lambda}(1+\varepsilon)\right)\right] \tag{22}$$

for all $\lambda>\lambda_0$ and all $0<u<v\leqslant 1-\delta$. Furthermore, it is straightforward to check that for every $\rho\in(0,1)$ we have

$$V\left(\rho,\frac{\rho+1}{2}\right)<\mathrm{e}^{1-\rho}.$$

Therefore,

$$\varepsilon=\frac{1}{2}\frac{\mathrm{e}^{1-\rho}-V(\rho,\frac{\rho+1}{2})}{\mathrm{e}^{1-\rho}+V(\rho,\frac{\rho+1}{2})}>0 \tag{23}$$

and

$$(1+\varepsilon)V\left(\rho,\frac{\rho+1}{2}\right)<(1-\varepsilon)\mathrm{e}^{1-\rho}. \tag{24}$$

By continuity of the function $V$ there is a $\delta=\delta(\rho)$ such that

$$(1+\varepsilon)V(u,v)<(1-\varepsilon)\mathrm{e}^{1-\rho} \tag{25}$$

for all $(u,v)$ in the set

$$q(\rho)=\left\{(u,v)\colon \rho-\delta<u<\rho,\ \frac{\rho+1}{2}-\delta<v<\frac{\rho+1}{2}+\delta\right\}. \tag{26}$$

Since $\rho<(\rho+1)/2$, we can and do choose a $\delta$ so small that $\rho-\delta>0$, $(\rho+1)/2+\delta<1$ and for every $(u,v)\in q(\rho)$ we have $u<v$. It follows by Eqs. (22) and (24) that there is a $\lambda_1>0$ such that for every $\lambda>\lambda_1$ and $(u,v)\in q(\rho)$ we have

$$U(b,c,\lambda)>\exp\left[-\frac{\lambda}{\log\lambda}(1-\varepsilon)\mathrm{e}^{1-\rho}\right]$$

with $\varepsilon$ defined by Eq. (23). Substituting this bound into Eq. (21) we obtain

$$P(L_\lambda(\boldsymbol{X})>1-\rho,\ N(1)=2)\geqslant\left(\mathrm{e}^{-1}\int_{(u,v)\in q(\rho)}\mathrm{d}u\,\mathrm{d}v\right)\exp\left[-\frac{\lambda}{\log\lambda}(1-\varepsilon)\mathrm{e}^{1-\rho}\right].$$

Denoting the positive constant in the right-hand side above by $c$ we obtain by Eq. (17) that

$$\frac{P(L_\lambda(\boldsymbol{X})>1-\rho,\ N(1)=2)}{H_\rho(\lambda)}\geqslant\frac{c}{\rho}\exp\left[\mathrm{e}^{1-\rho}\left(\frac{\lambda}{\log\lambda+(1-\rho)}-\frac{\lambda}{\log\lambda}(1-\varepsilon)\right)\right],$$

from which Eq. (19) follows.

**Remark.** It is clear that a construction similar to the above will work for a slightly more general class of infinitely divisible processes defined by

$$X(t)=\int_0^t \mathrm{e}^{\eta(s-t)}M(\mathrm{d}s), \tag{27}$$

$0\leqslant t\leqslant 1$, with the same infinitely divisible random measure $M$ as in Eq. (10), as long as $\eta>0$. In particular, the tail equivalence (9) fails without additional assumptions on the one-dimensional Lévy measure $\mu$ of $M$ (we will see in the following section what such additional assumptions may be). However, we will see in the last section of the paper that the process in Eq. (27) *always* satisfies Eq. (9) in the case $\eta<0$, without any additional assumptions on $\mu$.

## 3. Sample quantiles in the case of particularly heavy tails

The example in the previous section shows that subexponentiality of $H_\rho(\lambda)$ in Eq. (8) is not enough to guarantee the tail equivalence (9). Our goal in this section is to show that if, in addition to being subexponential, $H_\rho(\lambda)$ decays slowly enough (as measured in comparison with $H_*(\lambda)$ in Eq. (11)), then Eq. (9) holds.

The following theorem is the main result of this section.

**Theorem 1.** *Let $\boldsymbol{X}=(X(t),\ t\in T)$ be a measurable infinitely divisible process given by Eq.* (2). *Assume that*

$$\operatorname*{ess\,sup}_{t\in T}|X(t)|<\infty \ \textit{a.s.} \tag{28}$$

*Assume also that*

$$H_\rho(\lambda) \text{ is equivalent to the tail of a subexponential random variable} \tag{29}$$

*and that, furthermore, there is a function $u:(0,\infty)\to(0,\infty)$ such that*

$$\lim_{\lambda\to\infty}\frac{H_\rho\left(\lambda-bu(\lambda)\left(\frac{\log H_\rho(\lambda)}{\log|\log(H_\rho(\lambda))|}\right)^2\right)}{H_\rho(\lambda)}=1 \tag{30}$$

*for all $b>0$ and*

$$\lim_{\lambda\to\infty} \frac{(H_*(u(\lambda)))^2}{H_\rho(\lambda)} = 0. \tag{31}$$

*Then Eq. (9) holds.*

**Proof.** The lower bound in Eq. (9) follows just from the subexponentiality assumption (29), which implies the "long tail property" of $H_\rho(\lambda)$: for every $M>0$,

$$\lim_{\lambda\to\infty} \frac{H_\rho(\lambda+M)}{H_\rho(\lambda)} = 1.$$

See e.g. the proofs of Theorem 2.1 and Proposition 2.1 in Embrechts and Samorodnitsky (1995). Therefore, we only need to prove the upper-bound counterpart

$$\limsup_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X})>\lambda)}{H_\rho(\lambda)} \leqslant 1. \tag{32}$$

Furthermore, a standard splitting argument used in the above proofs shows that it is enough to prove the theorem in the compound Poisson case. Specifically, let $\boldsymbol{Y}_j = (Y_j(t), t\in T)$, $j=1,2,\ldots$ be a sequence of i.i.d. measurable stochastic processes, independent of a mean $\gamma$ Poisson random variable $N$. Let

$$\boldsymbol{X} = \sum_{j=1}^{N} \boldsymbol{Y}_j. \tag{33}$$

The assumptions of the theorem take, in this case, the following form. We assume, first of all, that

$$\operatorname*{ess\,sup}_{t\in T} |Y_1(t)| < \infty \text{ a.s.}. \tag{34}$$

Further, denote $\overline{F}(\lambda) = P(Q_\rho(\boldsymbol{Y}_1)>\lambda)$ $(=\gamma^{-1}H_\rho(\lambda))$. Assume that

$$\lim_{\lambda\to\infty} \frac{\overline{F}\left(\lambda - bu(\lambda)\left(\frac{\log \overline{F}(\lambda)}{\log|\log(\overline{F}(\lambda))|}\right)^2\right)}{\overline{F}(\lambda)} = 1 \tag{35}$$

for all $b>0$ and that

$$\lim_{\lambda\to\infty} \frac{P(\operatorname{ess\,sup}_{t\in T} |Y_1(t)| > u(\lambda))^2}{\overline{F}(\lambda)} = 0. \tag{36}$$

Then Eq. (32) holds, which takes, in the present case, the form of

$$\limsup_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X})>\lambda)}{\overline{F}(\lambda)} \leqslant \gamma. \tag{37}$$

We now prove Eq. (37). Let $f_1, f_2, \ldots$ be measurable functions from $T$ to $\mathbb{R}$. Starting with an obvious statement

$$Q_\rho(f_1+f_2) \leqslant Q_\rho(f_1) + \operatorname*{ess\,sup}_{t\in T} f_2(t),$$

one can easily check by induction that for every $k \geqslant 2$

$$Q_\rho\left(\sum_{j=1}^{k} f_j\right) \leqslant \sum_{j=1}^{k}(Q_\rho(f_j) \vee 0) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \min\left(\operatorname*{ess\,sup}_{t\in T}|f_i(t)|, \operatorname*{ess\,sup}_{t\in T}|f_j(t)|\right). \tag{38}$$

Furthermore, for every $\lambda > 0$,

$$\begin{aligned} P(Q_\rho(\boldsymbol{X}) > \lambda) &= \sum_{k=1}^{\infty} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(Q_\rho\left(\sum_{j=1}^{k}\boldsymbol{Y}_j\right) > \lambda\right) \\ &\leqslant \sum_{k=1}^{k_0(\lambda)-1} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(Q_\rho\left(\sum_{j=1}^{k}\boldsymbol{Y}_j\right) > \lambda\right) \\ &\quad + \sum_{k_0(\lambda)}^{\infty} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!} := I(\lambda) + II(\lambda), \end{aligned} \tag{39}$$

where

$$k_0(\lambda) = \left\lceil \frac{b|\log \overline{F}(\lambda)|}{\log|\log(\overline{F}(\lambda))|} \right\rceil. \tag{40}$$

We choose a

$$b > \max(2, e\gamma) \tag{41}$$

and we consider only those $\lambda$ for which $k_0(\lambda) \geqslant 2$.

Observe that for all $\lambda > \lambda_0(\gamma)$ we have by Eq. (41)

$$\begin{aligned} II(\lambda) &\leqslant 2\mathrm{e}^{-\gamma}\frac{\gamma^{k_0(\lambda)}}{(k_0(\lambda))!} \leqslant 2\mathrm{e}^{-\gamma}\frac{(e\gamma)^{k_0(\lambda)}}{(k_0(\lambda))^{k_0(\lambda)}} \\ &= 2\mathrm{e}^{-\gamma}\exp\left\{-k_0(\lambda)\log\frac{k_0(\lambda)}{e\gamma}\right\} \\ &\leqslant 2\mathrm{e}^{-\gamma}\exp\{-k_0(\lambda)(0.5\log|\log(\overline{F}(\lambda))|)\} \\ &\leqslant 2\mathrm{e}^{-\gamma}\exp\left\{-\frac{b}{2}|\log\overline{F}(\lambda)|\right\} = \mathrm{o}(P(Q_\rho(\boldsymbol{Y}_1) > \lambda)) \end{aligned} \tag{42}$$

as $\lambda \to \infty$.

Furthermore, by Eq. (38) we have

$$I(\lambda) \leqslant \sum_{k=1}^{k_0(\lambda)-1} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(\sum_{j=1}^{k}(Q_\rho(\boldsymbol{Y}_j) \vee 0) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\min\left(\operatorname*{ess\,sup}_{t\in T}|Y_i(t)|, \operatorname*{ess\,sup}_{t\in T}|Y_j(t)|\right) > \lambda\right)$$

$$\leqslant \sum_{k=1}^{k_0(\lambda)-1} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}\left[P\left(\sum_{j=1}^{k}(Q_\rho(\boldsymbol{Y}_j)\vee 0)>\lambda-\frac{k(k-1)}{2}u(\lambda)\right)\right.$$
$$\left.+P\left(\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\min\left(\operatorname*{ess\,sup}_{t\in T}|Y_i(t)|,\operatorname*{ess\,sup}_{t\in T}|Y_j(t)|\right)>\frac{k(k-1)}{2}u(\lambda)\right)\right]$$
$$\leqslant \sum_{k=1}^{k_0(\lambda)-1} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(\sum_{j=1}^{k}(Q_\rho(\boldsymbol{Y}_j)\vee 0)>\lambda-(k_0(\lambda)-1)^2u(\lambda)\right)$$
$$+\sum_{k=1}^{k_0(\lambda)-1} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}\frac{k(k-1)}{2}\left(P\left(\operatorname*{ess\,sup}_{t\in T}|Y_1(t)|>u(\lambda)\right)\right)^2$$
$$\leqslant \left(P\left(\operatorname*{ess\,sup}_{t\in T}|Y_1(t)|>u(\lambda)\right)\right)^2 E\left(\frac{N(N-1)}{2}\right)$$
$$+\sum_{k=1}^{\infty} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(\sum_{j=1}^{k}(Q_\rho(\boldsymbol{Y}_j)\vee 0)>\lambda-(k_0(\lambda)-1)^2u(\lambda)\right)$$
$$\leqslant \left(P\left(\operatorname*{ess\,sup}_{t\in T}|Y_1(t)|>u(\lambda)\right)\right)^2 E\left(\frac{N(N-1)}{2}\right)$$
$$+\sum_{k=1}^{\infty} \mathrm{e}^{-\gamma}\frac{\gamma^k}{k!}P\left(\sum_{j=1}^{k}(Q_\rho(\boldsymbol{Y}_j)\vee 0)>\lambda-b\,u(\lambda)\left(\frac{\log\overline{F}(\lambda)}{\log|\log(\overline{F}(\lambda))|}\right)^2\right). \quad (43)$$

Since $Q_\rho(\boldsymbol{Y}_1)$ is a subexponential random variable, it follows from Eqs. (36), (5) and (35) that

$$\limsup_{\lambda\to\infty}\frac{I(\lambda)}{\overline{F}(\lambda)}\leqslant\gamma, \quad (44)$$

and so the claim (37) follows from Eqs. (42) and (44). □

**Example 1** (*Intermediate regular variation*). Let $(X(t),\ t\in T)$ be given by

$$X(t)=\int_0^1 f(t,s)M(\mathrm{d}s), \quad (45)$$

where, once again, the Lévy measure $F$ of the infinitely divisible random measure $M$ is given by $F(\mathrm{d}s,\mathrm{d}x)=\mathrm{Leb}(\mathrm{d}s)\mu(\mathrm{d}x)$. We assume, for simplicity, that $\mu$ is a symmetric measure.

Let the tail of $\mu$ be *intermediate regular varying*. That is,

$$\lim_{r\downarrow 1}\liminf_{\lambda\to\infty}\frac{\mu(r\lambda,\infty)}{\mu(\lambda,\infty)}=1. \quad (46)$$

A consequence of intermediate regular variation is existence of numbers $0\leqslant\beta\leqslant\alpha<\infty$ and a $C>0$ such that for all $\lambda>1$

$$C^{-1}\lambda^{-\alpha}\leqslant\mu((\lambda,\infty))\leqslant C\lambda^{-\beta}. \quad (47)$$

The parameters $\alpha$ and $\beta$ are related to the so-called *Matuszewska indices* $\alpha_\mu$ and $\beta_\mu$ of $\mu$. (In fact, one can choose $\alpha=\alpha_\mu+\varepsilon$ and $\beta=\beta_\mu-\varepsilon$ if $\beta_\mu>0$, and $\beta=0$ if $\beta_\mu=0$, with $\varepsilon$ being an arbitrary positive number.) See for example Bingham et al. (1987), Cline (1994) and Cline and Samorodnitsky (1994) for more information on intermediate regular variation and related issues. In particular, intermediate regular variation implies subexponentiality.

To study sample quantiles of the process $\boldsymbol{X}$ defined by (45) we assume, for simplicity, that the kernel $f$ is uniformly bounded. That is, there is a finite constant $A$ such that $|f(t,s)|\leqslant A$ for all $t\in T$ and $0\leqslant s\leqslant 1$. However, this assumption can be greatly relaxed in different ways.

Assume that Eqs. (28) and (29) hold, and that

$$\alpha<2\beta. \tag{48}$$

Then Eq. (9) holds.

To prove this we only need to exhibit a function $u$ that satisfies both Eqs. (30) and (31) in Theorem 1. Observe that for the process (45) we have

$$\begin{aligned} H_\rho(\lambda) = & \int_0^1 \mathbf{1}(Q_\rho(\boldsymbol{f}(s))>0)\mu\left(\left(\frac{\lambda}{Q_\rho(\boldsymbol{f}(s))},\infty\right)\right)\mathrm{d}s \\ & +\int_0^1 \mathbf{1}(Q_\rho(-\boldsymbol{f}(s))>0)\mu\left(\left(-\infty,\frac{-\lambda}{Q_\rho(\boldsymbol{f}(s))}\right)\right)\mathrm{d}s \end{aligned} \tag{49}$$

and, similarly,

$$H_*(\lambda)=2\int_0^1 \mu\left(\left(\frac{\lambda}{\operatorname*{ess\,sup}_{t\in T}|f(t,s)|},\infty\right)\right)\mathrm{d}s. \tag{50}$$

Our first observation is that Eq. (49) can be viewed as the probability tail of a product of two independent random variables, one of which has an intermediate regular varying tail, and the other is bounded. We conclude by Theorem 3.4(ii) in Cline and Samorodnitsky (1994) that $H_\rho(\lambda)$ is itself intermediate regular varying.

It follows by Eq. (47) that for all $\lambda>A\vee 1$ we have

$$\begin{aligned} H_\rho(\lambda) \geqslant & C^{-1}\lambda^{-\alpha}\int_0^1 \mathbf{1}(Q_\rho(\boldsymbol{f}(s))>0)(Q_\rho(\boldsymbol{f}(s)))^\alpha\,\mathrm{d}s \\ & +C^{-1}\lambda^{-\alpha}\int_0^1 \mathbf{1}(Q_\rho(-\boldsymbol{f}(s))>0)(Q_\rho(-\boldsymbol{f}(s)))^\alpha\,\mathrm{d}s=c^{-1}\lambda^{-\alpha} \end{aligned}$$

for some $c>0$. Similarly, for all $\lambda>A\vee 1$ we have

$$H_*(\lambda)\leqslant c\lambda^{-\beta}.$$

We now choose $u(\lambda)=\lambda/(\log\lambda)^3$. Since $H_\rho(\lambda)\leqslant H_*(\lambda)$ we immediately conclude that for any $b>0$

$$\lambda - bu(\lambda)\left(\frac{\log H_\rho(\lambda)}{\log|\log(H_\rho(\lambda))|}\right)^2 \geqslant \lambda - b_1\frac{\lambda}{\log\lambda}$$

for some $b_1 > 0$ for all $\lambda$ large enough, and so Eq. (30) follows from the intermediate regular variation of $H_\rho(\lambda)$. Finally, Eq. (31) follows from Eq. (48).

The previous example shows that Theorem 1 allows us to treat situations more general than regular variation. However, this example still describes the case of a "power-like" tail decay. The following example shows that Theorem 1 may apply when the tails decay faster than any negative power of the level. It is also an example of an important distribution that fails to satisfy the conditions of Embrechts and Samorodnitsky (1995) but does satisfy the conditions of Theorem 1 above.

**Example 2.** Let us go back to the modified Ornstein–Uhlenbeck process of Section 2 that provided us with an example of a situation where Eq. (9) failed. Instead of Eq. (14) let us take this time the tail of $\mu$ being equal (or equivalent) to the tail of a *lognormal* distribution, i.e.

$$\mu((\lambda,\infty)) = 1 - \Phi\left(\frac{\log\lambda - a}{\sigma}\right), \tag{51}$$

where $a$ is a real number and $\sigma > 0$. It follows once again from Pitman (1980) that Eq. (51) is the tail of a subexponential random variable, and so $H_\rho$ is equivalent to the tail of a subexponential distribution as well. We claim that Eq. (9) holds in this case.

Once again, we only need to exhibit a function $u$ that satisfies both Eqs. (30) and (31) in Theorem 1. Choose

$$\frac{1}{\sqrt{2}} < \beta < 1 \tag{52}$$

and let $u(\lambda) = \lambda^\beta$. Then Eqs. (31) and (30) easily follow from Eq. (52).

We do not know the extent to which conditions (30) and (31) are the best possible. However, these two conditions should be viewed as quantifying the degree to which the tail $H_\rho(\lambda)$ decays at a slower rate than the tail $H_*(\lambda)$ does. We would like to mention that similar conditions occur naturally when one studies the tail of the product of independent subexponential random variables; see e.g. Cline and Samorodnitsky (1994).

## 4. Monotone kernels

The result in the previous section demonstrates that a slow enough rate of decay of $H_\rho(\lambda)$ in addition to subexponentiality of the latter implies the tail equivalence (9). This is true regardless of the structure of the kernel $f(t,s)$ in Eq. (2) (apart from the requirement that all integrals are well defined, the process is measurable and $H_\rho(\lambda)$ is subexponential). It turns out that another way to overcome the problems exposed in the counterexample of Section 2 is by imposing certain structural assumptions on the kernel. One such kind of structural assumptions is exhibited in this section.

**Theorem 2.** *Let* $X$ *be an infinitely divisible process given by Eq.* (2) (*with* $T$ *being an interval on the real line, and* $m$ *a Borel measure on* $T$), *such that for every* $s\in S$ *the function* $f(\cdot,s):T\to\mathbb{R}$ *is nonnegative and nondecreasing. Assume Eqs.* (28) *and* (29). *Then Eq.* (9) *holds.*

**Proof.** Once again, the standard splitting argument shows that it is enough to prove the theorem in the compound Poisson case, that is, in the case where the Lévy measure $F$ is a finite measure. We will thus consider a process (33), which we will write this time in a somewhat more explicit form. Let $N$ be a mean $\gamma=F(S\times\mathbb{R})$ Poisson random variable independent of a sequence of i.i.d. $S\times\mathbb{R}$-valued random vectors $((S_j,W_j),\ j\geqslant 1)$ with common distribution $\gamma^{-1}F$. We then consider

$$X(t)=\sum_{j=1}^{N} W_j f(t,S_j),\quad t\in T, \tag{53}$$

and we recall that we only need to prove Eq. (32). Marking the Poisson arrivals by the vectors $((S_j,W_j),\ j\geqslant 1)$ we can rewrite Eq. (53) in the form

$$X(t)=\sum_{j=1}^{N_+} W_j^+ f(t,S_j^+)-\sum_{j=1}^{N_-} W_j^- f(t,S_j^-),\quad t\in T, \tag{54}$$

where $N_+$ and $N_-$ are two independent Poisson random variables with means $\gamma_+=F(S\times(0,\infty))$ and $\gamma_-=F(S\times(-\infty,0))$ accordingly, independent of two independent sequences of i.i.d. $S\times\mathbb{R}$-valued random vectors $((S_j^+,W_j^+),\ j\geqslant 1)$ and $((S_j^-,W_j^-),\ j\geqslant 1)$ whose corresponding laws are given by

$$P((S_j^+,W_j^+)\in A)=\gamma_+^{-1}F(A\cap(S\times(0,\infty)))$$

and

$$P((S_j^-,W_j^-)\in A)=\gamma_-^{-1}F(A\cap(S\times(-\infty,0))).$$

Therefore,

$$Q_\rho(\boldsymbol{X})\leqslant Q_\rho\left(\left(\sum_{j=1}^{N_+} W_j^+ f(t,S_j^+),\quad t\in T\right)\right).$$

Now, for a fixed $0\leqslant\rho<1$ let

$$t_0=\inf\{t\in T:\ m(T\cap(-\infty,t]))>\rho\}.$$

There are two possibilities, $m(T\cap(-\infty,t_0])>\rho)$ and $m(T\cap(-\infty,t_0]))=\rho$. We consider the latter case; the treatment of the former one is similar.

Since the process $\boldsymbol{X}_+=(X_+(t)=\sum_{j=1}^{N_+} W_j^+ f(t,S_j^+),\ t\in T)$ is nondecreasing, we conclude that

$$Q_\rho(\boldsymbol{X}_+)=X_+(t_0+):=\lim_{t\downarrow t_0}X_+(t).$$

Therefore,

$$Q_\rho(\boldsymbol{X}) \leqslant Q_\rho(\boldsymbol{X}_+) = X_+(t_0+) = \sum_{j=1}^{N_+} W_j^+ f(t_0+, S_j^+). \tag{55}$$

Observe that for every $\lambda > 0$

$$\begin{aligned} P(W_j^+ f(t_0+, S_j^+) > \lambda) &= \gamma_+^{-1} F\{(s,x) \in S \times \mathbb{R}\colon xf(t_0+, s) > \lambda\} \\ &= \gamma_+^{-1} F\{(s,x) \in S \times \mathbb{R}\colon Q_\rho(x\boldsymbol{f}(s)) > \lambda\} = \gamma_+^{-1} H_\rho(\lambda). \end{aligned} \tag{56}$$

By the subexponentiality assumption (29) we may use Eqs. (55) and (5) to conclude that

$$\begin{aligned} \limsup_{\lambda \to \infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{H_\rho(\lambda)} &\leqslant \limsup_{\lambda \to \infty} \frac{P(\sum_{j=1}^{N_+} W_j^+ f(t_0+, S_j^+) > \lambda)}{H_\rho(\lambda)} \\ &= \gamma_+ \limsup_{\lambda \to \infty} \frac{P(W_j^+ f(t_0+, S_j^+) > \lambda)}{H_\rho(\lambda)} = 1. \end{aligned}$$

This proves Eq. (32) and so completes the proof of the theorem. □

It is obvious that the same argument will work if we assume that the function $f(\cdot, s): T \to \mathbb{R}$ is nonincreasing, instead of nondecreasing.

**Example 3.** Let $\boldsymbol{X} = (X(t),\, t \in [0,1])$ be a Lévy process with the Lévy measure $\mu$. That is, $\boldsymbol{X}$ is a process with stationary and independent increments, and it can be represented in the form

$$X(t) = \int_0^t M(\mathrm{d}s), \tag{57}$$

where the Lévy measure $F$ of the infinitely divisible random measure $M$ is given by $F(\mathrm{d}s, \mathrm{d}x) = \mathrm{Leb}(\mathrm{d}s)\mu(\mathrm{d}x)$. Assume that

$$\mu((\lambda, \infty)) \text{ is equivalent to the tail of a subexponential random variable.} \tag{58}$$

Then for every $0 \leqslant \rho < 1$

$$\lim_{\lambda \to \infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{\mu((\lambda, \infty))} = \rho. \tag{59}$$

Indeed, in this case

$$f(t,s) = \mathbf{1}(t > s), \quad t, s \in [0,1],$$

and

$$H_\rho(\lambda) = \rho\mu((\lambda, \infty)),$$

and so the assumptions of Theorem 2 are clearly satisfied. Therefore, Eq. (59) follows immediately.

The conclusion of this example can also be obtained from Dassios (1996). Indeed, the latter result shows that for any Lévy process

$$P(Q_\rho(\boldsymbol{X})>\lambda)=P(Z_1-Z_2>\lambda),$$

where $Z_1$ and $Z_2$ are independent non-negative random variables, such that

$$Z_1 \stackrel{\mathrm{d}}{=} \sup_{0\leqslant s\leqslant \rho} X(s).$$

Hence by, for example, the general result of Rosiński and Samorodnitsky (1993) one has

$$P(Q_\rho(\boldsymbol{X})>\lambda)\sim P(Z_1>\lambda)=P\left(\sup_{0\leqslant s\leqslant \rho} \boldsymbol{X}(s)>\lambda\right)\sim \rho\mu((\lambda,\infty))$$

as $\lambda\to\infty$.

**Example 4.** If $(X(t),\ t\in[0,1])$ is a Lévy motion of Example 3, then

$$Y(t)=\int_0^t \boldsymbol{X}(u)\,\mathrm{d}u, \quad t\in[0,1]$$

is the *integrated Lévy motion*. It can be represented in the form

$$Y(t)=\int_0^t (t-s)M(\mathrm{d}s), \quad t\in[0,1], \tag{60}$$

with $M$ as in the previous example. Assume that Eq. (58) holds. Then for every $0\leqslant\rho<1$

$$\lim_{\lambda\to\infty}\frac{P(Q_\rho(\boldsymbol{X})>\lambda)}{\int_0^\rho \mu((\lambda/(\rho-s),\infty))\,\mathrm{d}s}=1. \tag{61}$$

Indeed, we have

$$f(t,s)=\begin{cases}0 & \text{if } t\leqslant s,\\ t-s & \text{if } t>s,\end{cases}$$

and

$$H_\rho(\lambda)=\int_0^\rho \mu\left(\left(\frac{\lambda}{\rho-s},\infty\right)\right)\,\mathrm{d}s.$$

Now, the assumption (29) follows from Eq. (58) and Corollary 2.5 of Cline and Samorodnitsky (1994). Therefore, all the conditions of Theorem 2 are easily verified, and Eq. (61) follows.

**Example 5.** Let us consider once more the modified Ornstein–Uhlenbeck process of Section 2 in its general form (27). If $\eta<0$, then the kernel in the integral representation satisfies the conditions of Theorem 2 and Eq. (9) holds.

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